AN EXTENSION OF THE WORK OF V. GUILLEMIN ON COMPLEX POWERS AND ZETA FUNCTIONS OF ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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Abstract. The purpose of this note is to extend the results of Guillemin in [G] on elliptic self-adjoint pseudodifferential operators of order one, from operators defined on smooth functions on a closed manifold (scalar operators) to operators defined on smooth sections in a vector bundle of Hilbert modules of finite type over a finite von Neumann algebra.

0. Introduction

Let M be a closed Riemannian manifold of dimension m and E a vector bundle over M endowed with a hermitian metric. The fibers of E are finite dimensional vector spaces over $\mathbb{C}$ or, more general, finite type Hilbert modules over a von Neumann algebra $\mathcal{A}$. The first situation corresponds to the case $\mathcal{A} = \mathbb{C}$. Throughout this paper we will denote by $\Psi(E)$ or simply by Ψ the algebra of classical pseudodifferential operators acting on smooth sections in E (for the case when $\mathcal{A}$ is an arbitrary von Neumann algebra, see [BFKM] for definitions and properties). We will also denote by $\Psi^s(E)$ the subspace of pseudodifferential operators of complex order s. The total symbol $\sigma_{\text{total}}(x,\xi)$ of such an operator $A \in \Psi^s$ has locally an asymptotic expansion of the form:

$$\sigma_{\text{total}}(x,\xi) \sim \sum_{k=0}^{\infty} \sigma_{s-k}(x,\xi)$$

where $\sigma_{s-k}(x,\xi)$ are sections of the endomorphism bundle of the pull-back of E with respect to the projection map $T^*(M) \setminus \{0\} \to M$. Each section $\sigma_{s-k}(x,\xi)$ is a homogeneous function in the variable ξ of degree of homogeneity $s \in \mathbb{C}$, $\sigma_{s-k}(x,\lambda\xi) = \lambda^{s-k}(x,\xi)$ for any $\lambda > 0$.

The space $C^\infty(E)$ of smooth sections of E over M has a canonical metric

$$\langle f, g \rangle = \int_M \langle f(x), g(x) \rangle_x \, d\,vol$$

where $\langle \cdot\,, \cdot \rangle_x$ is the hermitian metric in the fibre above $x \in M$. The L^2 completion of $C^\infty(E)$ with respect to $\langle \cdot\,, \cdot \rangle$ will be denoted by $L^2(E)$. A pseudodifferential operator becomes an unbounded operator on $L^2(E)$.

We will consider now an elliptic pseudodifferential operator of order one $A \in \Psi^1$ which is self-adjoint and positive with respect to $\langle \cdot , \cdot \rangle$. Suppose that the spectrum of A is included in the interval (ϵ, ∞) for a sufficiently small $\epsilon > 0$. Then one can define the complex powers A^s, $s \in \mathbb{C}$ in the following way

$$A^s = \frac{1}{2\pi i} \int_\gamma \lambda^s (\lambda - A)^{-1}\, d\lambda \qquad \text{when } Re(s) < 0 \tag{0.1}$$

(where γ is a contour in the complex plane obtained by joining two lines parallel to the negative real axis by a circle around the origin) and

$$A^s = A^{s-k} A^k \qquad \text{for } Re(s) \geq 0 \tag{0.2}$$

for large enough $k \in \mathbb{Z}$ so that $s - k$ is negative.

One of the goals of our paper is to show that A^s is a pseudodifferential operator of complex order s. We remind the reader that this fact has been proven first by Seeley [S] in the case of finite dimensional hermitian bundle E and extended to the case of von Neumann bundles in [BFKM]. We will follow a different approach due to Guillemin [G]. In the same spirit of Guillemin, we will show that the zeta function of A defined as:

$$\zeta_A(s) = \mathrm{Trace}_N(A^s) \qquad \text{for } Re(s) < -m$$

has a meromorphic extension over the complex plane $\mathbb{C}$ with at most simple poles at $-m$, $-m+1$, ... The residue of ζ_A at $-m$ will be equal to a quantity that depens only on the principal symbol σ_1 of the operator A.

Guillemin treatment in [G] covers the case of pseudodifferential operators acting on smooth functions on M. We will extend his methods to the case of sections in the vector bundle E. The main difficulty arises from the fact that the algebra of endomorphisms of E is noncommutative (fiberwise it is equal to the algebra of the $\mathcal{A}-$invariant endomorphisms of the fiber, as compared to Guillemin's case where the fiber is canonically $\mathbb{C}$). Our paper has two main sections. In the first part we will show that A^s is a pseudodifferential operator of order $s \in \mathbb{C}$. The second part will be devoted to the zeta function of A.

Throughout the paper A will be a classical pseudodifferential operator of order 1. The case of an operator of any other positive order can be reduced to the case in which the order is equal to 1.

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1. Complex Powers of Pseudodifferential Operators

The goal of the section is proving the following:

Theorem 1.1. *Let A be a positive, self-adjoint pseudodifferential operator of order one. Suppose that $Spec(A) \in (\epsilon, \infty)$ for a sufficiently small $\epsilon > 0$. Then its complex powers A^s, defined as in (0.1) and (0.2), are pseudodifferential operators of order $s \in \mathbb{C}$.*

To show this we will need the following:

Proposition 1.2. *There exists a holomorphic family of pseudodifferential operators A_s for $s \in \mathbb{C}$ such that $A_0 = Id$, $A_s A_t = A_{s+t}$ and the difference $A_1 - A$ is a smoothing operator.*

$(A_s)_{s\in\mathbb{C}}$ can be thought of as an approximation of the powers of A that lie inside Ψ. We will show that $A_s - A^s$ are smoothing operators. Then Theorem 1.1 becomes a straightforward corollary of Proposition 1.2.

To construct the family $(A_s)_{s\in\mathbb{C}}$ it will be convenient to consider the cohomology of the group $(\mathbb{C}, +)$ with coefficients in the representation of $(\mathbb{C}, +)$ on the space of sections $C^\infty(\mathrm{End}(\tilde{E}))$. Here $\tilde{E}$ is the pull-back of the initial vector bundle E over M with respect to the projection map of the cosphere bundle $S^*(M) \to M$. This construction generalizes the cohomology considered by Guillemin in [G] for the trivial representation of $(\mathbb{C}, +)$ on the space of smooth functions on $S^*(M)$.

Let σ be a fixed section $\sigma : S^*(M) \to \mathrm{End}(\tilde{E})$ so that $\sigma(x, \xi) : E_x \to E_x$ is an invertible positive self-adjoint endomorphism for any $(x, \xi) \in S^*(M)$ (σ will be the restriction of the principal symbol of A to $S^*(M)$). The representation of $(\mathbb{C}, +)$ on $C^\infty(\mathrm{End}(\tilde{E}))$ we consider is the following one: any $s \in \mathbb{C}$ acts on a section $g : S^*(M) \to \mathrm{End}(\tilde{E})$ by $s \cdot g = \sigma^{-s} g \sigma^s$.

Let $\mathcal{C}^r = \mathcal{C}^r(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E}))$ be the space of functions

$$f : \underbrace{\mathbb{C} \times \mathbb{C} \times \ldots \mathbb{C}}_{r \text{ times}} \to C^\infty(\mathrm{End}(\tilde{E}))$$

that are smooth, $f(\cdot)(x, \xi) : \mathbb{C} \times \mathbb{C} \times \ldots \mathbb{C} \to \mathrm{End}(E_x)$ are holomorphic for any fixed $(x, \xi) \in S^*(M)$ and $f(s_1, \ldots, s_r) = 0$ if at least one s_i is equal to zero.

Let $\delta^r : \mathcal{C}^r \to \mathcal{C}^{r+1}$ defined as:

$$(\delta^r f)(s_0, s_1, \ldots, s_r) = s_0 \cdot f(s_1, \ldots, s_r) + \sum_{i=1}^{r} (-1)^i f(s_0, \ldots, s_{i-1} + s_i, \ldots, s_r) + (-1)^{r+1} f(s_0, \ldots, s_{r-1}).$$

Let $\mathcal{H}^r(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E})) = \mathrm{Ker}\,\delta^r / \mathrm{Im}\,\delta^{r-1}$.

Proposition 1.3. $\mathcal{H}^2(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E})) = 0$.

Moreover, for each 2-cocycle f there exists a unique 1-cochain h such that $\delta h = f$ and h has a prescribed value at 1, $h(1)$.

Proof. Let $f : \mathbb{C} \times \mathbb{C} \to C^\infty(\mathrm{End}(\tilde{E}))$ so that for all $a, b, c \in \mathbb{C}$

$$\begin{cases} f(0, b) = f(a, 0) = 0 \\ (\delta^2 f)(a, b, c) = a \cdot f(b, c) - f(a + b, c) + f(a, b + c) - f(a, b) = 0 \end{cases}$$

We will try to find $h : \mathbb{C} \to C^\infty(\mathrm{End}(\tilde{E}))$ such that

$$(\delta^1 h)(a, b) = \sigma^{-a} h(b) \sigma^a - h(a + b) + h(a) = f(a, b)$$

The existence of an h as above implies:

$$h'(a) = \sigma^{-a} h'(0) \sigma^a - \frac{\partial f}{\partial b}(a, 0) \tag{1.1}$$

Consider h to be the unique solution of the previous equation with $h(0) = 0$ and with a fixed prescribed value at 1, $h(1)$. h can be found in the following way:
Let $\Phi(t)$ be the automorphism of $C^\infty(\mathrm{End}(\tilde{E}))$ given by $A \to \sigma^{-t} A \sigma^t$. Then

$$h(a) = -\int_0^a \frac{\partial f}{\partial b}(t, 0)\, dt + \int_0^a \Phi(t)(h'(0))\, dt$$

If $T(a)A = \displaystyle\int_0^a \Phi(t) A\, dt$, then, in order to get any prescribed value for $h(1)$, we need to show that $T(1)$ is surjective. Indeed, we have:

$$\begin{aligned} T(1)A &= \int_0^{\frac{1}{2}} \sigma^{-t} A \sigma^t\, dt + \int_{\frac{1}{2}}^1 \sigma^{-t} A \sigma^t\, dt \\ &= T(\tfrac{1}{2})A + \Phi(\tfrac{1}{2}) T(\tfrac{1}{2})A = (\mathrm{Id} + \Phi(\tfrac{1}{2}))T(\tfrac{1}{2})A \end{aligned}$$

and by induction

$$T(1)A = (\mathrm{Id} + \Phi(\tfrac{1}{2}))(\mathrm{Id} + \Phi(\tfrac{1}{4})) \dots (\mathrm{Id} + \Phi(\tfrac{1}{2^n}))T(\tfrac{1}{2^n})A$$

But the map $A \to 2^n \displaystyle\int_0^{\frac{1}{2^n}} \sigma^{-t} A \sigma^t\, dt$ is close to the identity for a sufficiently large n so $T(\frac{1}{2^n})$ is invertible. $(\mathrm{Id} + \Phi(\frac{1}{2^i}))$ is invertible as well, because $\Phi(t)$ is positive self-adjoint for any real t.

Thus we obtain a continuous map $h : \mathbb{C} \to C^\infty(\mathrm{End}(\tilde{E}))$ that is holomorphic in all fibers $E_{(x,\xi)}$, $h \in \mathcal{C}^1$. We will show that $\delta\, h = f$ so f is a coboundary. To see this, let

$$g(a, b) = f(a, b) - (\sigma^{-a} h(b) \sigma^a - h(a + b) + h(a))$$

Clearly $\delta\, h = f$ if and only if $g \equiv 0$. Denote by $\dfrac{\partial}{\partial b}$ the partial derivative with respect to the second variable. Then:

$$\frac{\partial g}{\partial b}(a, b) = \frac{\partial f}{\partial b}(a, b) - \sigma^{-a} h'(b) \sigma^a + h'(a + b) \tag{1.2}$$

From (1.1) we get:

$$\begin{aligned} h'(b) &= \sigma^{-b} h(0) \sigma^b - \frac{\partial f}{\partial b}(b, 0) \quad \text{and} \\ h'(a + b) &= \sigma^{-(a+b)} h'(0) \sigma^{(a+b)} - \frac{\partial f}{\partial b}(a + b, 0) \end{aligned}$$

These two equalities and (1.2) imply

$$\begin{aligned} \frac{\partial g}{\partial b}(a, b) = \frac{\partial f}{\partial b}(a, b) - \sigma^{-a} &\left(\sigma^{-b} h'(0) \sigma^b - \frac{\partial f}{\partial b}(b, 0) \right) \sigma^a + \sigma^{-(a+b)} h'(0) \sigma^{(a+b)} - \\ &- \frac{\partial f}{\partial b}(a + b, 0) = \\ = \sigma^{-a} \frac{\partial f}{\partial b}&(b, 0) \sigma^a - \frac{\partial f}{\partial b}(a + b, 0) + \frac{\partial f}{\partial b}(a, b) = \\ = \frac{\partial}{\partial c} &\left[(\delta^2 f)(a, b, c) \right]_{|c=0} \end{aligned}$$

So $\dfrac{\partial g}{\partial b} = 0$ hence $g(a,b)$ is constant in b. When $b=0$ we have

$$g(a,0) = f(a,0) - \big(\sigma^{-a}h(0)\sigma^{a} - h(a) + h(a)\big) = 0$$

So $g \equiv 0$. Because f was chosen arbitrarily we conclude $\mathcal{H}^2(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E})) = 0$.
$\square$

We now proceed with the proof of Proposition 1.2. To show the existence of a family $(A_s)_{s\in\mathbb{C}}$ as stated in the proposition, we will show that there exists a family $(A_{(s)})_{s\in\mathbb{C}}$ of pseudodifferential operators that satisfies the conditions of Proposition 1.2 only up to smoothing operators. More precisely:

Proposition 1.4. *There exists a holomorphic family of pseudodifferential operators $(A_{(s)})_{s\in\mathbb{C}}$ with principal symbols $\sigma_{pr}(A_{(s)}) = (\sigma_{pr}(A))^s$ such that $A_{(0)} = Id$, $A_{(1)} \equiv A$ and $A_{(s)}A_{(t)} \equiv A_{(s+t)}$ modulo smoothing operators. This family is unique up to smoothing operators.*

Proof. The statement of the Theorem is equivalent to:

$$\begin{cases} A_{(s)}A_{(t)}A^{-1}_{(s+t)} \equiv Id & \pmod{\Psi^{-\infty}} \\ A_{(1)}A^{-1} \equiv Id & \pmod{\Psi^{-\infty}} \\ A_{(0)} = Id \end{cases} \tag{1.3}$$

(we denoted the space of smoothing operators by $\Psi^{-\infty}$)
To prove Proposition 1.4, we will construct $A_{(s)}$ inductively in $k \in \mathbb{N}$, such that

$$\begin{cases} A_{(s)}A_{(t)}A^{-1}_{(s+t)} \equiv Id & \pmod{\Psi^{-k}} \\ A_{(1)}A^{-1} \equiv Id & \pmod{\Psi^{-k}} \\ A_{(0)} = Id \end{cases} \tag{1.4}$$

For $k=1$ we can choose $(A_{(s)})_{s\in\mathbb{C}}$ to be a holomorphic family of pseudodifferential operators of order s with the principal symbol equal to σ^s where σ is the principal symbol of A. We can construct such a family using a partition of unity. Moreover $A_{(0)}$ can be chosen to be the identity. The operators $A_{(s)}A_{(t)}A^{-1}_{(s+t)}$ and $A_{(1)}A^{-1}$ are operators of order 0 with the principal symbol equal to the principal symbol of the identity. The relations (1.4) are satisfied modulo Ψ^{-1}.

Now suppose that the relations (1.4) hold for a certain $k \in \mathbb{N}$. We will construct a new family $(\tilde{A}_{(s)})_{s\in\mathbb{C}}$ that satisfies (1.4) for $k+1$, that is of the following form:

$$\tilde{A}_{(s)} = A_{(s)}(Id - H_{(s)}), \qquad H_{(s)} \in \Psi^{-k} \tag{1.5}$$

In this way $\tilde{A}_{(s)} - A_{(s)} \in \Psi^{s-k}$. We have:

$$\begin{aligned} \tilde{A}_{(s)}\tilde{A}_{(t)}\tilde{A}^{-1}_{(s+t)} &\equiv A_{(s)}(Id - H_{(s)})A_{(t)}(Id - H_{(t)})(Id + H_{(s+t)})A^{-1}_{(s+t)} \equiv \\ &\equiv A_{(s)}A_{(t)}A^{-1}_{(s+t)} - A_{(s)}H_{(s)}A_{(t)}A^{-1}_{(s+t)} - A_{(s)}A_{(t)}H_{(t)}A^{-1}_{(s+t)} + \\ &\quad + A_{(s)}A_{(t)}H_{(s+t)}A^{-1}_{(s+t)} \\ &\equiv Id + F_{(s,t)} - A_{(s)}H_{(s)}A_{(t)}A^{-1}_{(s+t)} - A_{(s)}A_{(t)}H_{(t)}A^{-1}_{(s+t)} + \\ &\quad + A_{(s)}A_{(t)}H_{(s+t)}A^{-1}_{(s+t)} \qquad \pmod{\Psi^{-k-1}} \end{aligned} \tag{1.6}$$

where $F_{(s,t)} = A_{(s)}A_{(t)}A^{-1}_{(s+t)} - Id$, $F_{(s,t)} \in \Psi^{-k}$ by the induction step. To proceed with the induction we have to find a family $(H_{(s)})_{s\in\mathbb{C}}$ that makes the right hand side of the equivalence (1.6) equal to the identity modulo Ψ^{-k-1}. If $\sigma_{pr}(F(s,t))$ and $h(s) = \sigma_{pr}(H(s))$ are the principal symbols, then the condition on $H(s)$ is equivalent to:

$$\begin{aligned} \sigma_{pr}(F(s,t)) &= \sigma^s h(s)\sigma^{-s} + \sigma^{s+t}h(t)\sigma^{-(s+t)} - \sigma^{s+t}h(s+t)\sigma^{-(s+t)} \quad \text{or} \\ \sigma^{-(s+t)}\sigma_{pr}(F(s,t))\sigma^{s+t} &= \sigma^{-t}h(s)\sigma^{t} - h(s+t) + h(t) \end{aligned} \tag{1.7}$$

Because both sides are sections in the bundle $\mathrm{End}(\tilde{E})$ over $T^*(M)\setminus\{0\}$ of degree of homogeneity $-k$, then the above equality is satisfied if it holds when both sections are restricted to the cosphere bundle $S^*(M)$. Let:

$$f(t,s) = \sigma^{-(s+t)}\sigma_{pr}(F(s,t))\sigma^{s+t} \qquad \text{restricted to } S^*(M) \tag{1.8}$$

We will show that $f \in \mathcal{C}^2(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E}))$ and $\delta^2 f = 0$. Then h as in (1.7) will be a 1-cochain so that $\delta h = f$.

We would also want the second condition of (1.4) to be satisfied so:

$$\begin{aligned} A^{-1}\tilde{A}_{(1)} &\equiv A^{-1}A_{(1)}(Id - H_{(1)}) \equiv \\ &\equiv Id + (A^{-1}A_{(1)} - Id) - A^{-1}A_{(1)}H_{(1)} \\ &\equiv Id \qquad (\mathrm{mod}\ \Psi^{-k-1}) \end{aligned}$$

and this holds if

$$h(1) = \sigma_{pr}(A^{-1}A_{(1)} - Id) \tag{1.9}$$

(we already know that $(A^{-1}A_{(1)} - Id) \in \Psi^{-k}$ from the induction step).

We will have to show that f is a cocycle in $\mathcal{C}^2$. Obviously, $f(0,t) = f(s,0) = 0$. We have:

$$\begin{aligned} (\delta^2 f)(s,t,r) &= \sigma^{-s}f(t,r)\sigma^s - f(s+t,r) + f(s,t+r) - f(s,t) = \\ &= \sigma^{-s}\left[\sigma^{-(t+r)}\sigma_{pr}(F(r,t))\sigma^{t+r}\right]\sigma^s - \sigma^{-(s+t+r)}\sigma_{pr}(F(r,s+t))\sigma^{s+t+r} \\ &\quad + \sigma^{-(s+t+r)}\sigma_{pr}(F(t+r,s))\sigma^{s+t+r} - \sigma^{-(s+t)}\sigma_{pr}(F(t,s))\sigma^{s+t} = 0 \end{aligned}$$

is equivalent to

$$\sigma_{pr}(F(r,t)) - \sigma_{pr}(F(r,s+t)) + \sigma_{pr}(F(t+r,s)) - \sigma^r\sigma_{pr}(F(t,s))\sigma^{-r} = 0 \tag{1.10}$$

To see this, consider the following equivalences modulo Ψ^{-k}:

$$\begin{aligned} &(Id + F(r,t))(Id + F(t+r,s))(Id - F(r,s+t))A_{(r)}(Id - F(t,s))A^{-1}_{(r)} \equiv \\ &\equiv A_{(r)}A_{(t)}A^{-1}_{(t+r)}A_{(t+r)}A_{(s)}A^{-1}_{(s+t+r)}A_{(s+t+r)}A^{-1}_{(s+t)}A^{-1}_{(r)}A_{(r)}A_{(s+t)}A^{-1}_{(s)}A^{-1}_{(t)}A^{-1}_{(r)} \equiv \\ &\equiv Id \end{aligned}$$

and the first term is also equivalent to

$$Id + F(r,t) - F(r,s+t) + F(t+r,s) - A_{(r)}F(t,s)A^{-1}_{(r)}$$

which proves (1.10). So $f(s,t) = \sigma^{-(s+t)}\sigma_{pr}(F(t,s))\sigma^{s+t}$ is a cocycle.

Proposition 1.3 provides us with a family $h(s)$ such that $\delta h = f$. We can choose this family so that (1.9) holds as well. This determines h in a unique way. If $(H_{(s)})_{s\in\mathbb{C}}$ is a holomorphic family of pseudodifferential operators of fixed order $-k$ with principal symbol $h(s)$ and $H_{(1)} = Id$, then $\tilde{A}_{(s)} = A_{(s)}(Id - H_{(s)})$ satisfies the equivalences (1.4) modulo Ψ^{-k-1}.

In this way we obtain a sequence of families of operators $(A^{(k)}_{(s)})_{s\in\mathbb{C}}$ that satisfy the relations (1.4) for each $k \in \mathbb{N}$. Moreover, $A^{(k+1)}_{(s)} - A^{(k)}_{(s)} \in \Psi^{s-k}$. Then, using a standard procedure as in Lemma 1.2.8 in [Gi], we can construct a family $(A_{(s)})_{s\in\mathbb{C}}$ whose asymptotic expansion of the total symbol will be equal to:

$$\sigma_{\text{total}}(A_{(s)}) \sim \sigma_{\text{total}}(A^{(1)}_{(s)}) + \sum_{k\geq 0} \sigma_{\text{total}}(A^{(k+1)}_{(s)} - A^{(k)}_{(s)})$$

The family $(A_{(s)})_{s\in\mathbb{C}}$ will satisfy the conditions of Proposition 1.4.

$(A_{(s)})_{s\in\mathbb{C}}$ is unique up to smoothing operators because it must satisfy the relations (1.4) for all $k \in \mathbb{N}$ and so it must be equal to $(A^{(k)}_{(s)})_{s\in\mathbb{C}}$ modulo Ψ^{-k}.

□

Proof of Theorem 1.1 and Proposition 1.2. Once we obtained the family of pseudodifferential operators $(A_{(s)})_{s\in\mathbb{C}}$, the proofs of Thm. 1.1 and Prop. 1.2 are identical to the proof of Theorem 5.1 in [G]. We can construct the one parameter group of operators as in Prop 1.2 using the differential equation:

$$\dot{A}_s = PA_s \qquad \text{with} \quad A_0 = Id$$

where $P = \dot{A}_{(0)}$. If $A_{(s)}$ is made a selfadjoint family in s (i.e. $A^*_{(s)} = A_{(\overline{s})}$) by replacing it with $\frac{1}{2}(A_{(s)} + A^*_{(\overline{s})})$, P becomes a selfadjoint operator. By construction $A_s \in \Psi^s$. Then, using a theorem of Stone (Thm VIII.7 and Thm VIII.8 [RS]), it can be shown that $A_s = (A_1)^s$ with P the infinitesimal generator of this one parameter group. In this case:

$$(A_1)^s - A^s = \frac{1}{2\pi i}\int_\gamma \lambda^s(\lambda - A)^{-1}(A - A_1)(\lambda - A_1)^{-1}\, d\lambda$$

and this is a smoothing operator. Because $(A_1)^s = A_s \in \Psi^s$ we obtain $A^s \in \Psi^s$.

□

2. Zeta Function of an Elliptic Pseudodifferential Operator

Let $(A_{(s)})_{s\in\mathbb{C}}$ be a family of pseudodifferential operators depending holomorphically on the complex parameter s, $A_{(s)} \in \Psi^s$. For $Re(s) < -dim(M)$, $A_{(s)}$ is a trace-class operator.

Definition 2.1. The trace function of the family$A_{(s)}$ is the holomorphic function $\text{Trace}_N\left(A_{(s)}\right)$ for $Re(s) < -dim(M)$.

The von Neumann trace of $A_{(s)}$ is obtained by integrating the von Neumann trace of the Schwartz kernel on M for $Re(s) < -dim(M)$. If A is an elliptic

positive self-adjoint pseudodifferential operator of order 1 with $Spec(A) \in (\epsilon, \infty)$ then its zeta function ζ_A is equal to the trace function associated with the family of its complex powers A^s.

In this section of our paper we will show that $\text{Trace}_N\left(A_{(s)}\right)$ has a meromorphic continuation to the whole complex plane with at most simple poles at $-m$, $-m+1$, $\ldots$, where $m = dim(M)$. This fact has been proved by Seeley [S]. Guillemin has a different proof in [G] that applies only for scalar pseudodifferential operators. We will adapt his proof for the case of operators that act on sections in a vector bundle E over the base space M.

We start by recalling some definitions and constructions in [G].

Let ω be the canonical symplectic form on the cotangent space $Y = T^*(M) \setminus \{0\}$. The multiplicative group $(\mathbb{R}^+, \cdot)$ acts on Y by multiplication along the fibre $(t,(x,\xi)) \xrightarrow{\rho} (x, t\xi)$. By identifying the groups $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$ via $\ln : \mathbb{R}^+ \to \mathbb{R}$, ρ can be seen as a 1-parameter group of isomorphisms. Let Ξ be the vector field on Y associated with this 1-parameter group and $\alpha = \iota_\Xi \, \omega$ be the contraction of ω along Ξ. Then the $(2m-1)$-form on Y, $\mu = \alpha \wedge \omega^{m-1}$, is homogeneous of degree m, $\rho_t^* \, \mu = t^m \mu$, and it is horizontal with respect to the fibration $Y = T^*(M) \setminus \{0\} \xrightarrow{\pi} S^*(M)$.

Let $\mathcal{B}$ be a von Neumann algebra. In our case, $\mathcal{B}$ will be the field of complex numbers $\mathbb{C}$, our initial von Neumann algebra $\mathcal{A}$ or $\text{End}_{\mathcal{A}}(V)$, where V is the generic fiber of the vector bundle $E \to M$. Let $\overline{\mathcal{P}}_s$ be the space of smooth homogeneous $\mathcal{B}$-valued functions defined on Y of degree of homogeneity $s \in \mathbb{C}$ and $\mathcal{P}_s$ the space of smooth scalar functions on Y of degree of homogeneity s. If $f \in \overline{\mathcal{P}}_{-m}$ then the $\mathcal{B}$-valued $(2m-1)$ form $f\mu$ is horizontal and invariant under the action of $(\mathbb{R}^+, \cdot)$ so it is of the form $\pi^* \mu_f$ where μ_f is a $(2m-1)$-form on $S^*(M)$.

Definition 2.2. The residue of $f \in \overline{\mathcal{P}}_{-m}$ is equal to the integral

$$\overline{\text{Res}}\, f = \int_{S^*(M)} \mu_f \quad \in \mathcal{B}$$

For $f \notin \overline{\mathcal{P}}_{-m}$ we define $\overline{\text{Res}}\, f = 0$.
If $\mathcal{B} = \mathbb{C}$, we will denote the residue simply by $\text{Res}\, f$.

Consider the Poisson bracket $\{\, , \,\}$ on $T^*(M)$ associated with the canonical symplectic form ω. Let $\{\mathcal{P}_s, \overline{\mathcal{P}}_t\}$ be the space of functions spanned by $\{f, g\}$ with $f \in \mathcal{P}_s$ and $g \in \overline{\mathcal{P}}_t$. Then $\{\mathcal{P}_s, \overline{\mathcal{P}}_t\} \subset \overline{\mathcal{P}}_{s+t-1}$. Following the same method as in [G] (Theorem 6.2), it can be shown that:

a) If $s \neq -m$ then $\{\mathcal{P}_1, \overline{\mathcal{P}}_s\} = \overline{\mathcal{P}}_s$.

b) If $s = -m$ then $\{\mathcal{P}_1, \overline{\mathcal{P}}_s\}$ consists of all functions f for which $\overline{\text{Res}}\, f = 0$.

Moreover, one can construct a family of functions $(g_i)_{i \in I}$, $g_i \in \mathcal{P}_1$ such that for any analytic family with parameter s, $f_s \in \overline{\mathcal{P}}_s$, defined on a strip $a - \epsilon \leq Im(s) \leq a + \epsilon$, $c \leq Re(s) \leq d$ for which $\overline{\text{Res}}\, f_{-m} = 0$, one can find $\delta \leq \epsilon$ and homogeneous functions $h_{i,s} \in \overline{\mathcal{P}}$ which are analytic in s on a narrower strip $a - \delta \leq Im(s) \leq a + \delta$, $c \leq Re(s) \leq d$, such that

$$f_s = \sum_{i \in I} \{g_i, h_{i,s}\}$$

(cf [G], Theorem 6.7)

Let us consider now a holomorphic family of pseudodifferential operators $(A_{(s)})_{s\in\mathbb{C}}$, $A_{(s)} \in \Psi^s$ and its associated trace function $\mathrm{Trace}_N\left(A_{(s)}\right)$. We define the residue of the family A to be $\mathrm{Res}\, A = \mathrm{Res}(\mathrm{Trace}_N\, \sigma_{pr}(A_{(-m)})) \in \mathbb{C}$.

We have the following theorem:

Theorem 2.3. *The trace function of the analytic family $(A_{(s)})_{s\in\mathbb{C}}$ has a meromorphic continuation to the whole complex plane with at most simple poles at $-m$, $-m+1$, $\dots$. The residue of $\mathrm{Trace}_N\left(A_{(s)}\right)$ at $s=-m$ is equal to*

$$\mathrm{res}_{|_{s=-m}}\,\mathrm{Trace}_N\left(A_{(s)}\right) = \gamma_0\,\mathrm{Res}\, A$$

where γ_0 is a constant depending only on $dim(M)$. For $A_{(s)} = A^s$ – the complex powers of an elliptic positive self-adjoint pseudodifferential operator of order one, the residue of the zeta function at $s=-m$ depends only on its principal symbol $\sigma = \sigma_{pr}(A)$ and is equal to $\gamma_0\,\mathrm{Res}(\sigma^{-m})$.

Proof. Let $(U_\alpha)_\alpha$ be an open cover of M with chosen trivializations of the vector bundle E over each U_α, $E_{|U_\alpha} \cong U_\alpha \times V$, with V the generic fiber. Using a partition of unity associated to the open cover $(U_\alpha)_\alpha$, we can write:

$$A_{(s)} = \sum_\alpha A_{\alpha\,(s)} + K_{(s)} \tag{2.1}$$

where $A_{\alpha\,(s)}$ are pseudodifferential operators of order s with support inside U_α and $K_{(s)}$ is a family of smoothing operators. Because the residue of the trace function of the family $A_{(s)}$ and $\mathrm{Res}\, A$ are both linear in A, it is sufficient to prove the theorem for $A_{\alpha\,(s)}$ and $K_{(s)}$. But $K_{(s)}$ is a family of smoothing operators and both the residues of their trace function and the residue $\mathrm{Res}\, K$ are zero. Thus we reduced the proof of the theorem to the case of one family $A_{(s)} = A_{\alpha\,(s)}$ supported in an open set $U = U_\alpha$ over which we have a trivialization of the vector bundle $\chi_\alpha : E_{|U} \to U \times V$. Moreover, because both the trace function $\mathrm{Trace}_N\left(A_{(s)}\right)$ and the residue $\mathrm{Res}\, A$ are obtained by integrating quantities that depend on the local expression of the total symbol of $A_{(s)}$, we can replace the bundle $E \to M$ with the trivial bundle $M \times V \to M$ and the operators $A_{\alpha\,(s)}$ with the pseudodifferential operators acting on sections of the trivial bundle $M \times V$ that are supported in the open set U_α and equal to $A_{\alpha\,(s)}$ via the isomorphism χ_α. To make things simple, we will denote this new family of operators by $A_{(s)}$ as well, and the new trivial bundle by E.

Following the ideas in [G], we consider the family $(s+m)A_{(s)}$. The principal symbol $(s+m)\sigma_{pr}(A_{(s)})$ can be represented by the $\mathcal{B}$-valued smooth homogeneous functions of degree s, $f_{(s)} : T^*(M) \setminus \{0\} \to \mathcal{B}$, with $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(V)$. For $s=-m$ we have $f=0$, so $\overline{\mathrm{Res}}\, f = 0$. Then there exist $\mathcal{B}$-valued functions $h^k_{(s)}$, $h^k_{(s)} \in \overline{\mathcal{P}}_s$ such that

$$f_{(s)} = \sum_k \{g_k, h^k_{(s)}\}$$

and $h^k_{(s)}$ are analytic on a strip $a-\epsilon \le Im(s) \le a+\epsilon$, $c \le Re(s) \le d$.

Let $G_k = G'_k \hat{\otimes} Id$ be a pseudodifferential operator acting on the space of sections $C^\infty(M)\hat{\otimes} V$ of the trivial bundle E with G'_k a scalar pseudodifferential operator that has the principal symbol equal to g_k and Id the identity operator. Let $(H^k_{(s)})$ be a

holomorphic family of pseudodifferential operators with the principal symbol equal to $h^k_{(s)}$. Then the principal symbol of the commutator is equal to

$$\sigma_{pr}\left[G_k, H^k_{(s)}\right] = \{g_k, h^k_{(s)}\}$$

so

$$(s+m)A_{(s)} = \sum_k \left[G_k, H^k_{(s)}\right] + B_{(s)} \qquad \text{with } B_{(s)} \in \Psi^{s-1}.$$

For $Re(s)$ sufficiently small, $\mathrm{Trace}_N\left([G_k, H^k_{(s)}]\right) = 0$, so $\mathrm{Trace}_N\left(A_{(s)}\right) = \dfrac{1}{s+m}\mathrm{Trace}_N\left(B_{(s)}\right)$ for $Re(s) < -m$. But $\dfrac{1}{s+m}\mathrm{Trace}_N\left(B_{(s)}\right)$ is a meromorphic function on the half-plane $Re(s) < -m+1$ with a simple pole at $s = -m$. So $\mathrm{Trace}_N\left(A_{(s)}\right)$ has a meromorphic extension to $Re(s) < -m+1$. Replacing the family $A_{(s)}$ by $B_{(s)}$ and using an induction argument, we can extend $\mathrm{Trace}_N\left(A_{(s)}\right)$ to a meromorphic function on the complex plane with at most simple poles at $-m$, $-m+1, \ldots$.

We will compare the residue of $\mathrm{Trace}_N\left(A_{(s)}\right)$ at $-m$ to the residue of the family $(A_{(s)})_{s\in\mathbb{C}}$, $\mathrm{Res}\, A = \mathrm{Res}(\mathrm{Trace}_N\, \sigma_{pr}(A_{(-m)}))$. Guillemin has showed ([G], Theorem 7.5) that in the scalar case there exists a constant γ_0 that depends only on the dimension of the manifold M such that

$$\mathrm{res}_{|s=-m}\, \mathrm{Trace}\, A = \gamma_0\, \mathrm{Res}\, A \tag{2.2}$$

We will extend this equality for the pseudodifferential operators acting on sections in the vector bundle E.

We will show a stronger equality:

$$\mathrm{res}_{|s=-m}\, \overline{\mathrm{Trace}}\, A = \gamma_0\, \overline{\mathrm{Res}}\, A_{(-m)} \tag{2.3}$$

where $(A_{(s)})_s$ is a holomorphic family of pseudodifferential operators acting on the sections of the trivial bundle $M \times V$, $\overline{\mathrm{Trace}}\, A_{(s)} = \int_M K_s(x,x)\, dx$ with $K_s(x,y)$ the Schwartz kernel of $A_{(s)}$, and $\overline{\mathrm{Res}}\, A_{(-m)} = \overline{\mathrm{Res}}\, \sigma_{pr}(A_{(-m)})$, both sides of the equality (2.3) being in the von Neumann algebra $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(V)$. The equality (2.2) will be then a direct consequence of (2.3) after passing to the von Neumann traces.

Both sides of the equality (2.3) depend only on the principal symbol of the operator $A_{(-m)}$. This is obvious for the right-hand side. If one considers another family $B_{(s)}$ with $\sigma_{pr}(B_{(-m)}) = \sigma_{pr}(A_{(-m)})$, then $(B_{(s)} - A_{(s)})$ is a family for which $\overline{\mathrm{Res}}\, \sigma_{pr}(B_{(-m)} - A_{(-m)}) = 0$, so, by a previous observation, $\overline{\mathrm{Trace}}(B_{(s)} - A_{(s)})$ has a meromorphic extension which is holomorphic at $s = -m$. So $\overline{\mathrm{Trace}}\, B_{(s)}$ and $\overline{\mathrm{Trace}}\, A_{(s)}$ will have the same residue at $s = -m$ and this shows that the left-hand side of (2.3) depends only on $\sigma_{pr}(A_{(-m)})$.

Both sides of (2.3), as functions of holomorphic families, will factor through the projection $A_{(s)} \to \sigma_{pr}(A_{(-m)}) \in \overline{\mathcal{P}}_{-m}$. It will be sufficient to show that the equality (2.3) holds on $\overline{\mathcal{P}}_{-m}$.

$\overline{\mathrm{Res}}$ vanishes exactly on $\{\mathcal{P}_1, \overline{\mathcal{P}}_{-m}\}$ and realizes a $\mathcal{B}$ isomorphism $\overline{\mathcal{P}}_{-m}/\{\mathcal{P}_1, \overline{\mathcal{P}}_{-m}\} \xrightarrow{\sim} \mathcal{B}$. For $f \in \{\mathcal{P}_1, \overline{\mathcal{P}}_{-m}\}$, $f = \sum\{g_k, h^k\}$, one can extend it to a holomorphic family of homogeneous symbols of degree of homogeneity $s \in \mathbb{C}$

by considering first the homogenous holomorphic extensions $h^k_{(s)} \in \overline{\mathcal{P}}_s$ and then taking $f_{(s)} = \sum\{g_k, h^k_{(s)}\}$. If $G_k = G'_k \hat{\otimes} Id$ is a pseudodifferential operator such that the scalar operator G'_k has the principal symbol equal to g_k and $(H^k_{(s)})$ is a holomorphic family of pseudodifferential operators with the principal symbol equal to $h^k_{(s)}$, then $A_{(s)}$ defined as $\sum[G_k, H^k_{(s)}]$ has the principal symbol at $s = -m$ equal to f and its trace is identically zero. This shows that $\operatorname{res}_{|s=-m} \overline{\operatorname{Trace}} A$ vanishes on $\{\mathcal{P}_1, \overline{\mathcal{P}}_{-m}\}$ as well. Because both $\operatorname{res}_{|s=-m} \overline{\operatorname{Trace}} A$ and $\overline{\operatorname{Res}} A_{(-m)}$ are $\mathcal{B}$ linear, one gets $\operatorname{res}_{|s=-m} \overline{\operatorname{Trace}} A = \overline{\operatorname{Res}} A_{(-m)} \cdot C$ with $C \in \mathcal{B}$.

Guillemin already showed this equality for a holomorphic family of scalar pseudodifferential operators $(A_{(s)})$ in which case C is a scalar constant γ_0. So $C = \gamma_0 \cdot \mathrm{Id}_{\mathcal{B}}$ and the equality (2.3) holds. Passing to the von Neuman trace, we get (2.2).

□

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